ON STABILITY AND EVOLUTION OF THE ELECTRIC CURRENT DISTRIBUTION IN A MEDIUM WITH NONLINEAR CONDUCTIVITY

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A. G. KULIKOVSKII and S. A. REGIRER
(Moscow)

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It is shown by an example of electrodynamic equations for a medium in which conductivity depends on the current density, that the change of type of three-dimensional operator in the system of equations with partial derivatives may lead to a nonevolutionary state of the system. The paper also discusses some physical factors which have not been taken into consideration in the equations and which slow down the rate of increase of perturbations, the corresponding changes in the system of equations, which describe the current distribution, and the possibility of obtaining steady-state solution.

Emets's paper [1] which recently appeared, discusses the two-dimensional stationary problem of the electric potential distribution for a medium which conductivity σ depends on the square of the current density. For a motionless medium this problem can be reduced to the solution of a quasilinear equation of the second order with respect to the z-component of the magnetic field B

$$(1 - \Phi B_{,x}^2)\ B_{,xx} + 2\Phi B_{,x} B_{,y} B_{,xy} + (1 - \Phi B^2_{,y})\, B_{,yy} = 0 \tag{1}$$

$$\Phi = \frac{\sigma' c^2}{8\pi^2 \sigma}, \quad \sigma' = \frac{d\sigma}{dj^2}, \quad \mathbf{j} = \frac{c}{4\pi} \operatorname{rot} \mathbf{e}_z B$$

As was shown in [1], Eq. (1) becomes hyperbolic if

$$1 - 2(\sigma'/\sigma) j^2 < 0 \tag{2}$$

i.e. at a sufficiently fast increase of the conductivity with the increase of the current. A similar phenomenon in a medium with anisotropic conductivity is mentioned in [2], which confirms the instability of the regimes corresponding to the hyperbolic region of Eqs. (1).

Let us consider the equations corresponding to the nonstationary processes. We shall limit our case by assuming that the perturbed current is in the same plane (x,y), and that j and B, as before, are independent of z. Ignoring displacement currents, we can write the following equation for B:

$$B_{,t} = (c^2 / 4\pi\sigma)\,[(1 - \Phi B_{,x}^2)\, B_{,xx} + 2\Phi B_{,x} B_{,y} B_{,xy} + (1 - \Phi B_{,y}^2) B_{,yy}] \tag{3}$$

To analyze the stability and evolution behavior of a certain solution it is necessary to linearize Eq. (3) and study the behavior of small perturbations. It is obvious that the coefficients of the higher derivatives for perturbations of the magnetic field are the same as in the stationary solution, with respect to which the linearization was carried out.

As the operator on the right-hand side of Eq. (3) is hyperbolic when condition (2) is satisfied, then we can choose the system of coordinates in such a way that the coefficients of the second derivatives with respect to x and y, will have opposite signs. In our case it would be sufficient to direst the axis y along the unperturbed current. In this case disturbances which depend only on x and t are described by an equation, principal terms of which are in turn equations of the thermal conductivity with the opposite sign of time.

It is obvious that such an equation is not evolutionary [3] and arbitrary small perturbations of the magnetic field can attain finite values in an arbitrary short time.

Therefore the steady regime described by Eq. (1) cannot be realized, for $1-2j^2\sigma'/\sigma<0$.

Owing to the rapid increase of perturbations, nonevolutionary equations cannot describe correctly changes of any physical quantity in time. Nonevolutionary solutions of the nonlinear equations in many cases can be regarded as a result of oversimplification in the derivation of these equations by discarding terms which are small for evolutionary solutions, but they can be essential for the perturbations which display a rapid increase. As the short wave disturbances increase most rapidly, then these could be the terms containing space derivatives of higher order or mixed derivatives with respect to space and time.

As an example of that what has been said, we shall consider equations describing the current distribution in a case when σ is the function of a quantity θ, which in turn obeys the equation of the type:

$$\varepsilon\theta_{,t} = \varkappa\Delta\theta + b(\theta_0 - \theta) + j^2/\sigma \tag{4}$$

where $\varkappa$, b are nonnegative functions of θ and j^2. The quantity θ can be for instance the temperature of "hot" current carriers in a plasma or in a solid. If the process is stationary and $\varkappa = 0$, then the function $\sigma(\theta)$ can be replaced in the equivalent way by the function $\sigma(j^2)$.

For simplicity sake we shall restrict our considerations to the case when an undisturbed solution is given by the equations $B = Ix$, $\theta = \text{const}$ and the perturbations depend only on x and t. Looking for the solution in the form $e^{ikx-\omega t}$ when the k are real, we can show that the roots of the dispersion equations λ_1 and λ_2 are real, and the greatest of them λ_1 vanishes when $k = 0$ together with the first derivative $d\lambda_1 / dk$ and $d^2\lambda_1/dk^2$ differs only in sign from the coefficient of B_{xx} in Eq. (3). Thus, Eq. (3) describes correctly the behavior of long wave perturbations. In the case when $1 - \Phi I^2 < 0$, the current distribution is unstable. The rate of increase of perturbations λ_1 is increasing at first with the rise of k^2, and then at $\varkappa \neq 0$ begins to decrease so that $\lambda_1 \to -\infty$ at $k^2 \to \infty$. In this way the thermal conductivity leads to a damping of the short wave perturbations. If $\varkappa = 0$, then k^2 increases with the increase of λ_1 but when $k^2 \to \infty$ it tends to a finite limit. In this case the rate of increase of perturbations is limited, as the finite value of the material thermal capacity ε prevents very rapid increase of perturbations. It is interesting to note that if the displacement currents are taken into account in the electrodynamic equations we shall find that this leads also to limitation of the rate of increase of perturbations for $\sigma = \sigma(j^2)$, in spite of the fact that in this case λ is a very large quantity when k^2 is large.

When the thermal conductivity is taken into account in Eq. (4), one can expect to find stationary solutions for the current distribution, each of these solutions at small values of $\varkappa$ is close to a certain solution of Eq. (1) where it is elliptical and differs substantially from it in the regions where the condition of ellipticity is violated. Within these regions the quantity $\Delta\theta$ should be of the order of $1/\varkappa$, so that the product $\varkappa\Delta\theta$ is comparable with other terms of the equation.

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SOME POSSIBLE MOTIONS OF A HEAVY SOLID IN A FLUID

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V. N. RUBANOVSKII
(Moscow)

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We consider the motion of a heavy solid with internal cyclical motions in a heavy ideal fluid of infinite extent under the conditions that the weight of the body and the Archimedean buoyant force form a couple, and that the impulsive force is vertical (the Chaplygin condition [1]).

Three new special cases in which the equations of motion of the above mechanical system are integrable [1 – 5] are considered. The equations in these cases admit of a system of three linear particular solutions. It is shown that all of these particular solutions are expressible in terms of elliptic functions of time, and that the rotational portion of the motions of the solid in the fluid described by these particular solutions is similar to the motion of a balanced gyrostat [6].

Algebraic solutions containing two arbitrary constants are given by Clebsch's second and third cases of integrability of the Kirchhoff-Clebsch equations [2 and 3] of the classical problem of internal motion of a solid bounded by a simply connected surface through an ideal fluid of infinite extent in all directions. These algebraic solutions immediately yield the "complete set" of four first integrals required for reduction of the problem to quadratures.

Liapunov [7] noted that Clebsch's third case of integrability could be considered as a certain limiting case of his second case. The fourth first integrals for these Clebsch cases are represented in a single form.

The fourth integrals in the classical cases of Steklov and Liapunov were reduced to a single form by Kolosov [8] and Kharlamov [9 and 10].

1. We consider the problem of motion in an unbounded ideal homogeneous imcompressible fluid of a heavy solid bounded by a simply connected surface with multiply connected cavities filled completely with an ideal fluid engaged in nonvortical motion. The Chaplygin conditions [1] apply, i.e. the weight of the body and the fluid in its cavities and the Archimedean buoyant force form a couple. We assume that the motion of the boundless fluid due to the motion of the solid in it is nonvortical and that the fluid is at rest at infinity.